Stability of stochastic differential equations driven by variants of stable processes

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Abstract

In this paper we investigate two variants of stable processes, namely tempered stable subordinators and modified tempered stable process as well as their renormalization. We study the weak convergence in the Skorohod space and prove that they satisfy the uniform tightness condition. Finally, applications to the stability of SDEs driven by these processes are discussed.

1 Introduction

In the last decade, Lévy processes have received a great deal of attention fuelled by numerous financial applications, see Cont and Tankov [7], for an introduction to some financial models driven by Lévy processes. In this paper we study the stability of stochastic differential equations (SDEs) driven by one parameter family of Lévy processes, namely two variants of stable processes.

Firstly, we consider the classes of tempered stable subordinators X_α^{TSS} and modified tempered stable processes X_α^{MTS} with $\alpha \in (0, 1/2)$ and prove the weak convergence, in the Skorohod space endowed with the Skorohod topology, of X_α^{TSS} (resp. X_α^{MTS}) to the gamma process when $\alpha \to 0$ (resp. the normal inverse Gaussian process X^{NIG} when $\alpha \to 1/2$). The family $\{X_\alpha^{MTS}, \alpha \in (0, 1/2)\}$ was considered in [20] to develop the GARCH option price model. We want also to point out that the weak convergence of X_α^{TSS} to the gamma process was first established in [25] using other considerations. Indeed it is proved that the gamma process has been obtained as weak limit of renormalized stable processes. The family of the renormalized stable processes we identify as the family of tempered stable processes $\{X_\alpha^{TSS}, \alpha \in (0, 1/2)\}$, see Remark 2.2 below. Moreover, we would like also to mention the work of Rydberg [21] where an approximation of the NIG process, based on an appropriate discretization of the Lévy measure, was discussed. In this paper, instead we use the modified tempered stable process as an approximation of the NIG process.

The stability problem consists in investigating the conditions under which the solutions converge weakly. However, it is well known that the weak convergence is not sufficient to ensure the conver-

gence of stochastic integral , see [16] and references therein. Among the sufficient conditions we cite the uniform tightness (**UT**), introduced by Striker [24]. It should be noted that this condition has been used extensively to establish the results of stability of stochastic differential equations since its introduction, see for example [10], [12], [14], [16] and [17]. Thus we show that both driven families $\{X_\alpha^{TSS},\, X_\alpha^{MTS},\, \alpha \in (0,1/2)\}$ satisfy the (**UT**) condition. This allows us to establish the stability result of SDEs driven by these families.

Secondly, it is proven in [2] that the standard Brownian motion $\{W(t),\, t \in [0,1]\}$ is obtained as a weak limit, in the Skorohod space equipped with the uniform metric, of a suitable renormalization of certain classes of Lévy process which includes the family $\{X_\alpha^{MTS},\, \alpha \in (0,1/2\}$. More precisely, the Brownian motion W can be approximated by an appropriate renormalization of the compensated sum of small jumps of a given Lévy process, see Proposition 3.5 below. In the same spirit we mention the work [8] which completes, in some sense the previous one, where it is shown that the process $\{t,\, t \in [0,1]\}$ is a weak limit of a renormalized (in an appropriate sense) sum of small jumps of classes of subordinator. We note that the family $\{X_\alpha^{TSS},\, \alpha \in (0,1/2\}$ is among those classes. These two results lead us to consider the stability problem of SDEs driven by these renormalized processes. The main tools we use to prove the stability result are the uniform tightness of the renormalized families and the stability of SDEs established in [17].

2 Lévy processes and infinite divisibility

We start by recalling a few well-known facts about infinitely divisible distributions. We consider a class of Borel measures on $\mathbb{R}$ satisfying the following conditions:

$$\Lambda(\{0\}) = 0, \tag{1}$$

$$\int_{-\infty}^{+\infty} (s^2 \wedge 1) d\Lambda(s) < \infty. \tag{2}$$

This class will be denoted by $\mathfrak{M}$.

De Finetti [9] introduced the notion of an infinitely divisible distribution and showed that they have an intimate relationship with Lévy processes. By the Lévy-Kintchine formula, all infinitely divisible distributions F_Λ are described via their characteristic function:

$$\phi_\Lambda(u) = \int_{-\infty}^{+\infty} e^{iux} dF_\Lambda(x) = e^{\Psi_\Lambda(u)},\ u \in \mathbb{R},$$

where the characteristic exponent Ψ_Λ, is given as

$$\Psi_\Lambda(u) = ibu - \frac{1}{2}cu^2 + \int_{-\infty}^{+\infty} (e^{ius} - 1 - ius\, \mathbb{1}_{\{|s|<1\}}(s))\, d\Lambda(s),$$

where $b \in \mathbb{R}, c \geq 0$.

We assume as given a filtered probability space $(\Omega,\, \mathcal{F},\, P,\, (\mathcal{F}_t)_{t\in[0,1]})$ satisfying the usual hypothesis. A Lévy process $X = \{X(t), t \in [0,1]\}$ has the property

$$\mathbb{E}(e^{iuX(t)}) = e^{t\Psi(u)},\ t \in [0,1]\,,\ u \in \mathbb{R},$$

where $\Psi(u)$ is the characteristic exponent of $X(1)$ which has an infinitely divisible distribution. Thus, any infinitely divisible distribution F_Λ generates in a natural way a Lévy process X by setting the law of $X(1)$, $\mathcal{L}(X(1)) = F_\Lambda$. The three quantities (b, c, Λ) determine the law $\mathcal{L}(X(1))$. Since the distribution of a Lévy process $X = \{X(t), t \in [0,1]\}$ is completely determined by the marginal distribution $\mathcal{L}(X(1))$, and thus the process X itself completely. The measure Λ is called the Lévy measure whereas (b, c, Λ) is called the Lévy-Khintchine triplet.

Let us now give some examples of Lévy processes which will be used later on. We will present three classes related to the sample path properties. Namely subordinators, processes with paths of finite and infinite variations.

2.1 Subordinators

A subordinator is a one-dimensional increasing Lévy process starting from 0. Subordinators form one of the simplest family of Lévy processes. The law of a subordinator is specified by the Laplace transform of its one dimensional distributions. We assume throughout this paper that these processes have no drift.

We consider a subclass in $\mathfrak{M}$ of measures supported on $\mathbb{R}_+$ satisfying the following

$$\Lambda(0,\infty) = \infty, \tag{3}$$

$$\int_0^1 s\, d\Lambda(s) < \infty. \tag{4}$$

Any Lévy measure Λ satisfying conditions (3) and (4) generates a subordinator X, see for example [6, Theorem 1.2]. We can therefore give its Laplace transform

$$\psi_\Lambda(u) := \mathbb{E}(e^{-uX(1)}) = \exp\left(\int_0^\infty (e^{-su} - 1)\, d\Lambda(s)\right), \quad u \in \mathbb{R}_+.$$

Remark 2.1. **(i)** *When X is a subordinator, the Laplace transform of its marginal distributions is much more useful, for both theoretical and practical applications, than the characteristic function.*

(ii) *The assumption (3) implies that the process X has infinite activity, that is, almost all paths have infinitely many jumps along any time interval of finite length. Whereas the condition (4) guarantees that almost all paths of X have finite variation.*

Examples

1. **Gamma process**. Consider the Lévy measure Λ_γ with density with respect to the Lebesgue measure defined by

$$d\Lambda_\gamma(s) := \frac{e^{-s}}{s} \mathbb{1}_{\{s>0\}} ds.$$

Then the corresponding process is known as gamma process. A simple calculation shows that

$$\psi_{\Lambda_\gamma}(u) = \frac{1}{1+u},$$

and the Laplace transform of the corresponding process has the form

$$\mathbb{E}_{\mu_\gamma}\left(e^{-uX_\gamma(t)}\right) = \exp\left(-t\log(1+u)\right) = \frac{1}{(1+u)^t},\ t \in [0,1].$$

Here μ_γ denotes the law of $X_\gamma(1)$.

2. **Stable subordinator (SS)**. Let $\alpha \in (0,1)$ be given and let Λ_α^{SS} be the Lévy measure given by

$$d\Lambda_\alpha^{SS}(s) := \frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{s^{1+\alpha}}1\!\!1_{\{s>0\}}ds.$$

Then we have

$$\psi_\Lambda(u) = \exp\left(-u^\alpha\right),$$

and

$$\mathbb{E}_{\mu_\alpha^{SS}}\left(e^{-uX_\alpha^{SS}(t)}\right) = \exp\left(-tu^\alpha\right),\ t \in [0,1].$$

Here μ_α^{SS} denotes the law of $X_\alpha^{SS}(1)$.

3. **Tempered stable subordinator (TSS).** A tempered stable subordinator is obtained by taking a stable subordinator and multiplying the Lévy measure by an exponential function, that is, an exponentially tempered version of the stable subordinator. More precisely, for $\alpha \in (0,1)$, we consider the Lévy measure

$$d\Lambda_\alpha^{TSS}(s) := \frac{1}{\alpha}e^{-s}d\Lambda_\alpha^{SS}(s) = \frac{1}{\Gamma(1-\alpha)}\frac{e^{-s}}{s^{1+\alpha}}1\!\!1_{\{s>0\}}ds. \tag{5}$$

Then we have

$$\psi_{\Lambda_\alpha^{TSS}}(u) = \exp\left(\frac{1-(1+u)^\alpha}{\alpha}\right)$$

and

$$\mathbb{E}_{\mu_\alpha^{TSS}}\left(e^{-uX_\alpha^{TSS}(t)}\right) = \exp\left(-t\frac{1-(1+u)^\alpha}{\alpha}\right),\ t \in [0,1].$$

Now let us give a concrete realization of a subordinator due to Tsilevich-Vershik-Yor [25]. We denote by

$$D = \left\{\eta = \sum z_i\delta_{x_i},\ x_i \in [0,1],\ z_i \in \mathbb{R}_+,\ \sum |z_i| < \infty\right\}$$

the real linear space of all finite real discrete measures in $[0,1]$. We define the coordinate process $\{X(t), t \in [0,1]\}$ on D by

$$X(t) : D \longrightarrow \mathbb{R}_+,\ \eta \mapsto X(t)(\eta) := \eta([0,t]), \quad t \in [0,1]$$

and $\mathcal{F}_t := \sigma(X(s),\ s \le t)$ denotes its own filtration.

Let Λ be a Lévy measure satisfying conditions (3) and (4) and μ_Λ be a probability measure on $(D, \mathcal{F}_1)$ with Laplace transform given by

$$\mathbb{E}_{\mu_\Lambda}\left(\exp\left(-\int_0^1 f(t)\,d\eta(t)\right)\right) = \exp\left(\int_0^1 \log(\psi_\Lambda(f(t))\,dt\right).$$

Here f is an arbitrary non-negative bounded Borel function on $[0,1]$. In particular, when $f(s) = u1\!\!1_{[0,t]}(s)$, $u > 0$, $t \in (0,1]$ the Laplace transform of $X(t)$ is given by

$$\mathbb{E}_{\mu_\Lambda}(e^{-uX(t)}) = \exp\left(t\log(\psi_\Lambda(u))\right),\ t \in [0,1].$$

We call the pair (X, μ_Λ) a realization of a Lévy process with Lévy measure Λ which is a subordinator, cf. [25, Remark 2.1].

Now we would like to highlight the link between tempered stable and stable subordinators. First of all it follows from (5) that the Lévy measures Λ_α^{TSS} and Λ_α^{SS} are equivalent. Then we obtain from [23, Theorem 33.1] that X_α^{SS} and X_α^{TSS} have equivalent laws with density given in [23, Theorem 33.2], see (6) below. We notice that the authors in [25, 26] constructed a family of measures, equivalent to α-stable laws with given densities which converges weakly to the gamma measure. This is the content of the following remark.

Remark 2.2. *Let $\tilde{X}_\alpha$ be a process such that the law $\tilde{\mu}_\alpha := \mathcal{L}(\tilde{X}_\alpha(1))$ is equivalent to μ_α^{SS} with density*

$$\frac{d\tilde{\mu}_\alpha}{d\mu_\alpha^{SS}}(\eta) = \frac{\exp(-\alpha^{-1/\alpha}X(1)(\eta))}{\mathbb{E}_{\mu_\alpha^{SS}}\left(\exp(-\alpha^{-1/\alpha}X(1)(\eta))\right)} = e^{\alpha^{-1}}e^{-\alpha^{-1/\alpha}X(1)(\eta)}. \tag{6}$$

Then the law of the tempered stable subordinator X_α^{TSS} is nothing but the law of the process $\alpha^{-1/\alpha}\tilde{X}_\alpha$.

2.2 Lévy processes with finite variation paths

We consider a Lévy process with the following triplet $(0,0,\Lambda)$. We are interested here in the subclass of $\mathfrak{M}$ satisfying

$$\Lambda(\mathbb{R}) = \infty, \tag{7}$$

$$\int_{|s|\leq 1} |s|\ d\Lambda(s) < +\infty. \tag{8}$$

Condition (8) means that the corresponding Lévy process has finite variation paths.

Examples

1. **Stable process (S)**. Symmetric α-stable processes X_α^S, with $\alpha \in (0,1)$, are the class of Lévy processes whose characteristic exponents correspond to those of symmetric α-stable distributions. The corresponding Lévy measure is given by

$$d\Lambda_\alpha^S(s) = \left(\frac{1}{|s|^{1+\alpha}}1\!\!1_{\{s<0\}} + \frac{1}{s^{1+\alpha}}1\!\!1_{\{s>0\}}\right)ds.$$

The characteristic exponent $\Psi_{\Lambda_\alpha^S}$ has the form

$$\Psi_{\Lambda_\alpha^S}(u) = -|u|^\alpha,\ u \in \mathbb{R}.$$

2. **Tempered stable process (TS)**. It is well known that α-stable distributions, with $\alpha \in (0,1)$, have infinite p-th moments for all $p \geq \alpha$. This is due to the fact that its Lévy density decays polynomially. Tempering the tails with the exponential rate is one choice to ensure finite moments. The tempered stable distribution is then obtained by taking a symmetric α-stable distribution and multiplying its Lévy measure by an exponential functions on each half of the real axis. In explicit

$$d\Lambda_\alpha^{TS}(s) = \left(\frac{e^{-|s|}}{|s|^{1+\alpha}} 1\!\!1_{\{s<0\}} + \frac{e^{-s}}{s^{1+\alpha}} 1\!\!1_{\{s>0\}} \right) ds.$$

The characteristic exponent $\Psi_{\Lambda_\alpha^{TS}}$ is given by

$$\Psi_{\Lambda_\alpha^{TS}}(u) = \Gamma(-\alpha)[(1-iu)^\alpha + (1+iu)^\alpha - 2], \; u \in \mathbb{R}.$$

The associated Lévy process will be called tempered stable process and denoted by X_α^{TS}.

3. **Modified tempered stable process (MTS).**

The MTS distribution is obtained by taking an α-stable law with $\alpha \in (0, 1/2)$ and multiplying the Lévy measure by a modified Bessel function of the second kind on each side of the real axis. It is infinitely divisible and has finite moments of all orders. It behaves asymptotically like the 2α-stable distribution near zero and like the TS distribution on the tail. Then the Lévy density is given by

$$d\Lambda_\alpha^{MTS}(s) = \frac{1}{\pi} \left(\frac{K_{\alpha+\frac{1}{2}}(|s|)}{|s|^{\alpha+\frac{1}{2}}} 1\!\!1_{\{s<0\}} + \frac{K_{\alpha+\frac{1}{2}}(s)}{s^{\alpha+\frac{1}{2}}} 1\!\!1_{\{s>0\}} \right) ds.$$

$K_{\alpha+\frac{1}{2}}$ is the modified Bessel function of the second kind given by the following integral representation

$$K_{\alpha+\frac{1}{2}}(s) = \frac{1}{2} \left(\frac{s}{2}\right)^{\alpha+\frac{1}{2}} \int_0^{+\infty} \exp\left(-t - \frac{s^2}{4t}\right) t^{-\alpha-\frac{3}{2}}\, dt. \tag{9}$$

The characteristic exponent has the form

$$\Psi_{\Lambda_\alpha^{MTS}}(u) = \frac{1}{\sqrt{\pi}} 2^{-\alpha-\frac{1}{2}}\, \Gamma(-\alpha)[(1+u^2)^\alpha - 1], \; u \in \mathbb{R}.$$

The induced Lévy process, denoted by X_α^{MTS}, will be called modified tempered stable process. For additional details on MTS distributions the reader may consult [20].

2.3 Lévy process of infinite variation paths

Finally, we would like to consider a subclass of $\mathfrak{M}$ satisfying (7) and the following condition

$$\int_{|s|\leq 1} |s| \; d\Lambda(s) = \infty. \tag{10}$$

Examples

1. Symmetric α-stable processes, tempered stable processes, with $\alpha \in (1,2)$ and modified tempered stable processes, with $\alpha \in (1/2, 1)$.

2. **Normal inverse Gaussian process (NIG).** The NIG distribution was introduced in finance by Barndorff-Nielsen. It might be of interest to know that the NIG distribution is a special case of the generalized hyperbolic distribution, introduced also by Barndorff-Nielsen to model the logarithm of particle size, see references below.

   Let $\{X^{NIG}(t),\, t \in [0,1]\}$ be a Lévy process with Lévy measure given by

$$d\Lambda^{NIG}(s) = \frac{K_1(|s|)}{\pi|s|} ds,$$

   where K_1 is the modified Bessel function of the second kind with index 1. The characteristic exponent is equal to

$$\Psi_{\Lambda^{\mathrm{NIG}}}(u) = \left(1 - \sqrt{1+u^2}\right), u \in \mathbb{R}.$$

   The process $\{X^{NIG}(t),\, t \in [0,1]\}$ is a Lévy process with the triplet $(0, 0, \Lambda^{NIG})$.

   For further results related to the normal inverse Gaussian distributions see Barndorff-Nielsen [3, 4] and Rydberg [21, 22].

We conclude this section with the following remark.

Remark 2.3. *All Lévy processes considered before are such that:*

1. *Their paths belong to the set of all càdlàg functions, denoted by* $\mathbb{D}([0,1], \mathbb{R})$*, i.e. all real-valued right continuous with left limits functions on* $[0,1]$*.*

2. *They are pure jump semimartingales processes without fixed times of discontinuity.*

3 Weak convergence and uniform tightness

In this section at first we present a result on weak convergence of the above families in $\mathbb{D}([0,1], \mathbb{R})$ endowed with the Skorohod topology $\mathcal{J}_1$, $(\mathbb{D},\, \mathcal{J}_1)$. This convergence will be denoted by "$\xrightarrow{\mathbb{D}}$". On the other hand, since we will deal with continuous limit processes, we are interested in the tightness and weak convergence in the space $\mathbb{D}([0,1], \mathbb{R})$ equipped with the uniform topology $\mathcal{U}$, $(\mathbb{D}, \mathcal{U})$. We will denoted them by "$\mathbb{C}$-tight" and "$\xrightarrow{\mathbb{C}}$", respectively. Finally, after recalling the definition of the uniform tightness as well as a useful criterion, we prove that the processes considered satisfy this condition, cf. Propositions 3.10 and 3.11 below.

3.1 Weak convergence in $(\mathbb{D}, \mathcal{J}_1)$

In this subsection we present the weak convergence in $(\mathbb{D}, \mathcal{J}_1)$ of the families of processes $\{X_\alpha^{TSS}, \alpha \in (0, 1/2)\}$ and $\{X_\alpha^{MSS},\, \alpha \in (0, 1/2)\}$. We start with the following elementary lemma.

Lemma 3.1. *We have the following weak convergence of the one dimensional law:*

(i) $X_\alpha^{TSS}(1) \xrightarrow{\mathcal{L}} X_\gamma(1)$, $\alpha \to 0$.

(ii) $X_\alpha^{MTS}(1) \xrightarrow{\mathcal{L}} X^{NIG}(1)$, $\alpha \to 1/2$.

Proof. The result in (i) is a consequence of Proposition 6.3 in [25].
(ii) It is easy to see that the characteristic exponent $\Psi_{\Lambda_\alpha^{MTS}}(1)$ converge to $\Psi_{\Lambda^{NIG}}(1)$ when α goes to 1/2. This implies that $X_\alpha^{MTS}(1)$ converge weakly to $X^{NIG}(1)$. $\square$

Proposition 3.2. *We have the following weak convergence in* $(\mathbb{D}, \mathcal{J}_1)$*:*

(i) $X_\alpha^{TSS} \xrightarrow{\mathbb{D}} X_\gamma$, *as* $\alpha \to 0$.

(ii) $X_\alpha^{MTS} \xrightarrow{\mathbb{D}} X^{NIG}$, *as* $\alpha \to 1/2$.

Proof. Since Lévy processes are semimartingales with stationary independent increments, then it follows from [13, Corollary 3.6] that the convergence of the marginal laws of $X_\alpha^{TSS}(1)$ and $X_\alpha^{MTS}(1)$ is equivalent to the weak convergence of processes X_α^{TSS} and X_α^{MTS} in $(\mathbb{D}, \mathcal{J}_1)$. $\square$

Now we are interested in the weak convergence of certain renormalization of pure jump subordinator. Let X be a subordinator with Lévy measure Λ satisfying the conditions (3)-(4) and X_ε be the sum of its jumps of size in $(0, \varepsilon)$. Then the corresponding Lévy measure Λ_ε is nothing but the restriction of Λ to $(0, \varepsilon]$. We denote the expectation of $X_\varepsilon(1)$ by $\mu(\varepsilon) := \int_{(0,\varepsilon]} s d\Lambda(s)$. We consider the renormalized process $Y_\varepsilon := \mu(\varepsilon)^{-1} X_\varepsilon$ and state the following convergence result proved in [8].

Proposition 3.3. *The following statements hold, as* $\varepsilon \to 0$.

(i) *If* $\mu(\varepsilon)/\varepsilon \to c$, *where* $0 < c < +\infty$, *then* $Y_\varepsilon \xrightarrow{\mathbb{D}} c^{-1} X_c^*$ *where* X_c^* *is a pure jump subordinator with Lévy measure given by* $d\Lambda_c^*(s) = \mathbb{1}_{(0,1]}(s)(c/s)ds$.

(ii) *If* $\mu(\varepsilon)/\varepsilon \to +\infty$, *then* $Y_\varepsilon \xrightarrow{\mathbb{D}} \mathbf{t} := \{t, t \in [0,1]\}$.

Remark 3.4. *Since* Y_ε *are Lévy processes and the limit process in the statement (ii) is continuous, then it follows from [19, Theorem 19] that the convergence holds also in* $(\mathbb{D}, \mathcal{U})$ *as follows*

(ii)' *If* $\mu(\varepsilon)/\varepsilon \to +\infty$, *then* $Y_\varepsilon \xrightarrow{\mathbb{C}} \mathbf{t}$.

We give some examples of Lévy processes which illustrate the above proposition.

1. Gamma process, $\mu(\varepsilon)/\varepsilon \to 1$.
2. Stable and tempered stable subordinators, $\alpha \in (0,1)$, $\mu(\varepsilon)/\varepsilon \to +\infty$.

3.2 Weak convergence in $(\mathbb{D}, \mathcal{U})$

In this subsection we are interested in the weak convergence of certain renormalizations of Lévy processes. Let X be a Lévy process with characteristic function of the form

$$\mathbb{E}(e^{iuX(t)}) = \exp\left(ibu - \frac{1}{2}cu^2 + \int_{-\infty}^{+\infty} (e^{ius} - 1 - ius\, 1\!\!1_{\{|s|<1\}}(s))\, d\Lambda(s) \right)$$

where $t \in [0,1]$, $u \in \mathbb{R}$ and the Lévy measure Λ does not have atoms in some neighbourhood of the origin. For each $\varepsilon \in (0,1)$, let us consider $\tilde{X}_\varepsilon$ the compensated sum of jumps of X taking values in $(-\varepsilon, \varepsilon)$. It is well known that $\{\tilde{X}_\varepsilon,\, 0 < \varepsilon \leq 1\}$ is a family of Lévy processes with characteristic function

$$\mathbb{E}(e^{iu\tilde{X}_\varepsilon(t)}) = \exp\left(t \int_{|s|\leq\varepsilon} (e^{ius} - 1 - ius)\, d\Lambda(s) \right), \quad t \in [0,1].$$

It is clear that, for each $\varepsilon > 0$, $\tilde{X}_\varepsilon$ is a martingale with jumps bounded by ε with $\mathbb{E}(\tilde{X}_\varepsilon(1)) = 0$ and

$$\mathbb{E}(\tilde{X}_\varepsilon^2(1)) = \int_{|s|\leq\varepsilon} s^2 d\Lambda(s) =: \sigma^2(\varepsilon).$$

We consider the renormalization process $\tilde{Y}_\varepsilon := \sigma(\varepsilon)^{-1}\tilde{X}_\varepsilon$ and state the following convergence result due to Asmussen and Rosiński [2].

Proposition 3.5. *The following are equivalent*

1. $\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$ *as* $\varepsilon \to 0$, *where* W *is a standard Brownian motion.*
2. $\dfrac{\sigma(\varepsilon)}{\varepsilon} \longrightarrow \infty$ *as* $\varepsilon \to 0$.

Remark 3.6. *For each* $\varepsilon \in (0,1)$, $\tilde{Y}_\varepsilon$ *is a Lévy process with characteristic function given by*

$$\mathbb{E}(e^{iu\tilde{Y}_\varepsilon(t)}) = \exp\left(t \left[iub_\varepsilon + \int_{\mathbb{R}} \left(e^{ius} - 1 - ius 1\!\!1_{\{|s|\leq 1\}}(s) \right) d\tilde{\Lambda}_\varepsilon(s) \right] \right),\ t \in [0,1],$$

where the Lévy measure $\tilde{\Lambda}_\varepsilon$ *is defined, for any* $B \in \mathcal{B}(\mathbb{R})$, *by*

$$\tilde{\Lambda}_\varepsilon(B) := \Lambda(\sigma(\varepsilon)B \cap (-\varepsilon, \varepsilon)), \tag{11}$$

and

$$b_\varepsilon := -\sigma(\varepsilon)^{-1} \int_{\sigma(\varepsilon)\wedge\varepsilon \leq |s| \leq \varepsilon} s\, d\Lambda(s). \tag{12}$$

We give some examples of Lévy processes for which the above renormalization converge.

1. Symmetric α-stable processes, $\alpha \in (0,2)$, $\sigma(\varepsilon) = (2/(2-\alpha))^{1/2}\varepsilon^{1-\alpha/2}$.
2. Tempered stable processes, $\alpha \in (0,1)$, $\sigma(\varepsilon) \geq (2/(2-\alpha))^{1/2}\varepsilon^{1-\alpha/2}e^{-\varepsilon/2}$.
3. Modified tempered stable processes, $\alpha \in (0,1/2)$, $\sigma(\varepsilon) \approx (2/((2-2\alpha)\pi))^{1/2}\varepsilon^{1-\alpha}$.
4. Normal inverse Gaussian, $\sigma(\varepsilon) \approx (2/\pi)^{1/2}\varepsilon^{1/2}$.

We notice that the examples 1. and 4. above were considered in [2].

3.3 Uniform tightness of Lévy processes

First we recall the definition and criterion of the uniform tightness (**UT**) needed later on. The following definition was proposed by Jakubowski, Mémin and Pagès [14].

Definition 3.7. *A sequence of semimartingales $\{Z^n,\, n \geq 1\}$ is said to be uniformly tight if for each $t \in (0,1]$, the set*

$$\left\{ \int_0^t H^n(s^-)\, dZ^n(s), H^n \in \mathcal{H}, n \geq 1 \right\}$$

is stochastically bounded (uniformly in n).

In the above definition $\mathcal{H}$ denotes the collection of simple predictable processes of the form

$$H(t) = H_0 + \sum_{i=1}^{m} H_i \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where H_i is $\mathcal{F}_{t_i}$-measurable such that $|H_i| \leq 1$ and $0 = t_0 \leq \ldots \leq t_{m+1} = t$ is a finite partition of $[0,t]$.

In practice it is not easy to verify the (**UT**) condition as stated in Definition 3.7. Thus we look for a more convenient criterion due to Kurtz and Protter [16]. Let Z be an adapted process with càdlàg paths and $\{Z^n, n \in \mathbb{N}\}$ be a sequence of semimartingales, with the canonical decompositions

$$Z^n(t) = M^n(t) + A^n(t), \tag{13}$$

where A^n is a predictable process with locally bounded variation and M^n is a (locally bounded) local martingale.

Proposition 3.8. *[cf. [16]] Assume that $Z^n \xrightarrow{\mathbb{D}} Z$ and one of the following two conditions holds*

$$\sup_{n\in\mathbb{N}} \left\{ \mathbb{E}\left([M^n, M^n](1) + \int_0^1 |dA^n(t)| \right) \right\} < +\infty, \tag{14}$$

$$\sup_{n\in\mathbb{N}} \left\{ \mathbb{E}\left(\sup_{t\leq 1} |\Delta M^n(t)| + \int_0^1 |dA^n(t)| \right) \right\} < +\infty. \tag{15}$$

*Then $\{Z^n, n \in \mathbb{N}\}$ satisfies (**UT**).*

Remark 3.9. 1. *If Z is a continuous semimartingale then we assume that $Z^n \xrightarrow{\mathbb{C}} Z$.*

2. *The conditions (14) and (15) imply the uniform controlled variation (**UCV**) of $\{Z^n, n \in \mathbb{N}\}$ introduced in [16].*

3. *Since $Z^n \xrightarrow{\mathbb{D}} Z$ then the (**UT**) and (**UCV**) are equivalent, see [16].*

Next, we are interested in the decomposition (13) for a Lévy process Z. We start by splitting Z into two parts depending on the size of the jumps:

$$Z(t) = R(t) + N(t)$$

with $N(t) = \sum_{s\le t} \Delta Z(s)\ \mathbb{1}_{\{|\Delta Z(s)|>1\}}$ and R with jumps bounded by 1. Since R is a Lévy process with bounded jumps its canonical decomposition is, by means of [1, pp. 103], of the simple form $R(t) = R_0(t) + t\mathbb{E}(R(1))$ where $\{R_0(t) : t \in [0,1]\}$ is a càdlàg centred square-integrable martingale with jumps bounded by 1. Hence the decomposition (13) takes the form

$$Z(t) = R_0(t) + t\mathbb{E}(R(1)) + \sum_{s\le t} \Delta Z(s)\ \mathbb{1}_{\{|\Delta Z(s)|>1\}}. \tag{16}$$

Now we are able to state the main result of this subsection.

Proposition 3.10. *The following families satisfy (**UT**)*

(i) $\{X_\alpha^{TSS},\ \alpha \in (0,1/2)\}$,

(ii) $\{X_\alpha^{MTS},\ \alpha \in (0,1/2)\}$.

Proof. Since the families $\{X_\alpha^{TSS},\ \alpha \in (0,1/2)\}$ and $\{X_\alpha^{MTS},\ \alpha \in (0,1/2)\}$ are weakly convergent, then in order to obtain the (**UT**) property, we have only to check condition (14) of Proposition 3.8.

(i) The decomposition (16) for the process X_α^{TSS} is given by

$$X_\alpha^{TSS}(t) = R_{\alpha,0}^{TSS}(t) + t\mathbb{E}(R_\alpha^{TSS}(1)) + \sum_{s\le t} \Delta X_\alpha^{TSS} \mathbb{1}_{\{|\Delta X_\alpha^{TSS}|>1\}}. \tag{17}$$

Thus, condition (14) becomes

$$\sup_{\alpha\in(0,1/2)} \left(\int_0^1 s^2\ d\Lambda_\alpha^{TSS}(s) + \int_0^{+\infty} s\ d\Lambda_\alpha^{TSS}(s) \right) < +\infty,$$

which is simple to verify.

(ii) It is easy to see that $\mathbb{E}(R_\alpha^{MTS}(1)) = 0$. Then the (**UT**) condition follows from

$$\sup_{\alpha\in(0,1/2)} \left(\int_{|s|\le 1} s^2\ d\Lambda_\alpha^{MTS}(s) + \int_{|s|>1} |s|\ d\Lambda_\alpha^{MTS}(s) \right) < +\infty.$$

To show this we use the integral representation (10) for the Bessel function $K_{\alpha+1/2}$ and estimate the above integrals as

$$\begin{aligned}
\int_{|s|>1} |s|\ d\Lambda_\alpha^{MTS}(s) &= 2^{-\alpha-1/2} \int_1^{+\infty} \int_0^{+\infty} s\ e^{-\frac{s^2}{4t}} e^{-t} t^{-(\alpha+3/2)} dt ds \\
&= 2^{1/2-\alpha} \int_0^{+\infty} e^{-(t+\frac{1}{4t})} t^{-\alpha-1/2} dt \\
&\le 5\ 2^{1/2-\alpha} \int_{1/4}^{+\infty} e^{-(t+\frac{1}{4t})} dt.
\end{aligned}$$

$$\begin{aligned}
\int_{|s|\le 1} s^2\ d\Lambda_\alpha^{MTS}(s) &\le 2 \int_0^{+\infty} s^2\ d\Lambda_\alpha^{MTS}(s) \\
&= \sqrt{\pi}\ 2^{1/2-\alpha}\ \Gamma(1-\alpha).
\end{aligned}$$

This completes the proof. $\square$

Next we state the (**UT**) property for the renormalized families $\{Y_\varepsilon,\, \varepsilon \in (0,1)\}$ and $\{\tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$.

Proposition 3.11. **(i)** *Assume that $\mu(\varepsilon)/\varepsilon$ converges in $(0,+\infty]$. Then the renormalized family $\{Y_\varepsilon,\, \varepsilon \in (0,1)\}$ satisfies (**UT**).*

(ii) *Assume that $\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$. Then the renormalized family $\{\tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$ satisfies (**UT**).*

Proof. (i) Since the process Y_ε is a pure jump subordinator, then the condition (14) becomes

$$\sup_{\varepsilon\in(0,1)} \left\{ \mathbb{E}\left(\int_0^1 |dY_\varepsilon(t)| \right) \right\} = \sup_{\varepsilon\in(0,1)} \mathbb{E}\left(Y_\varepsilon(1)\right) = 1. \tag{18}$$

So the (**UT**) condition is a consequence of Proposition 3.8.
(ii) First notice that, for each $\varepsilon \in (0,1)$, $\tilde{Y}_\varepsilon$ is a martingale with jumps bounded by $\varepsilon/\sigma(\varepsilon)$. Thus we obtain

$$\mathbb{E}\left(\sup_{t\le 1} |\Delta \tilde{Y}_\varepsilon(t)| \right) \le \frac{\varepsilon}{\sigma(\varepsilon)}.$$

As a consequence of statement 2 of Proposition 3.5 we have

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left(\sup_{t\le 1} |\Delta \tilde{Y}_\varepsilon(t)| \right) < \infty,$$

which implies that condition (15) is satisfied. Since $\tilde{Y}_\varepsilon$ is weakly convergent, then (**UT**) condition follows from Proposition 3.8. $\square$

4 Stability of stochastic differential equation driven by Lévy processes

The previous section established the weak convergence and uniform tightness for certain families of Lévy processes. Now we would like to apply these results to study the stability problem for SDEs driven by these families of Lévy processes. For a survey on SDEs driven by Lévy processes we refer to [5]. To begin, we give some notations useful in the sequel: for each $n \in \{2,3,\ldots\}$, "$\xrightarrow{\mathbb{D}^n}$" and "$\mathbb{D}^n$-tight" denote the weak convergence and tightness in $\mathbb{D}([0,1],\mathbb{R}^n)$ endowed with the Skorohod topology. In the same way "$\xrightarrow{\mathbb{C}^n}$" and "$\mathbb{C}^n$-tight" denote the weak convergence and tightness for the uniform topology.

4.1 The modified tempered stable case

We will make the following assumptions

(H.1) $a_\alpha, h_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous such that $|a_\alpha(x)| + |h_\alpha(x)| \le K(1+|x|)$ for all $\alpha \in (0,1/2)$, $x \in \mathbb{R}$.

(H.2) The family a_α (resp. h_α) converge uniformly to a (resp. h) on each compact set in $\mathbb{R}$, as $\alpha \to 0$.

We consider the following SDEs

$$dY_\alpha^{TSS}(t) = a_\alpha(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t) + h_\alpha(Y_\alpha^{TSS}(t))dt, \quad Y_\alpha^{TSS}(0) = 0, \tag{19}$$

and

$$dY(t) = a(Y(t))dX_\gamma(t) + h(Y(t))dt, \quad Y(0) = 0. \tag{20}$$

Remark 4.1. *1. Under the assumption **(H.1)**, for each $\alpha \in (0, 1/2)$, the equation (19) admits a weak solution, see Jacod and Mémin [11].*

2. Since the coefficients a_α and a are not Lipschitz, then we do not have uniqueness of solutions for either equation (19) or equation (20).

The first stability result concerns the class of tempered stable subordinators.

Theorem 4.2. *Under the assumptions **(H.1)-(H.2)** we have*

1. *The family of processes $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight.*
2. *Any limit point (Y, X_γ) of the family $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ satisfies equation (20).*
3. *If uniqueness in law holds for the equation (20), then*

$$(Y_\alpha^{TSS}, X_\alpha^{TSS}) \xrightarrow{\mathbb{D}^2} (Y, X_\gamma), \quad \alpha \to 0.$$

Proof. 1. At first we show that the family $\{Y_\alpha^{TSS},\ \alpha \in (0, 1/2)\}$ verify the (**UT**) condition. Under assumption **(H.1)** and the uniform tightness of the family $\{X_\alpha^{TSS},\ \alpha \in (0, 1/2)\}$ we can show, using a Gronwall type inequality (see [18, Lemme 29-1]), that the family $\{\sup_{s\in[0,1]} |Y_\alpha^{TSS}(s)|,\ \alpha \in (0, 1/2)\}$ is bounded in probability. Hence the family $\{\sup_{s\in[0,1]} |a_\alpha(Y_\alpha^{TSS}(s))|,\ \alpha \in (0, 1/2)\}$ (resp. $\{\sup_{s\in[0,1]} |h_\alpha(Y_\alpha^{TSS}(s))|,\ \alpha \in (0, 1/2)\}$) is also bounded in probability since a_α (resp. h_α) has at most linear growth. Therefore it is easy to see that the family $\{\int_0^\cdot h_\alpha(Y_\alpha^{TSS}(t))dt,\ \alpha \in (0, 1/2)\}$ satisfies the (**UT**) condition. On the other hand, the uniform tightness of the family $\{\int_0^\cdot a_\alpha(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t), \alpha \in (0, 1/2)\}$ follows from [17, Lemme 1-6]. As a consequence we get the (**UT**) condition for the family $\{Y_\alpha^{TSS},\ \alpha \in (0, 1/2)\}$.

On the next step we show that the family of processes $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight. Since the function a_α is continuous we can always find a sequence of C^2 functions, $\{a_{\alpha,n},\ n \in \mathbb{N}\}$, which approximate uniformly a_α on compact sets of $\mathbb{R}$. Now let us consider the sequence of process $Y_{\alpha,n}^{TSS}$ defined by

$$dY_{\alpha,n}^{TSS}(t) = a_{\alpha,n}(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t) + h_\alpha(Y_\alpha^{TSS}(t))dt, \quad Y_\alpha^{TSS}(0) = 0. \tag{21}$$

As the function $a_{\alpha,n}$ is of class C^2 then we get from [17, Lemme 1-7] that the family $\{a_{\alpha,n}(Y_\alpha^{TSS}), \alpha \in (0, 1/2)\}$ is uniformly tight. Now it follows from [17, Proposition 3-3] (see also [15]) that the family of processes $(\int_0^\cdot a_{\alpha,n}(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t), X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight and consequently $(Y_{\alpha,n}^{TSS}, X_\alpha^{TSS})$ is also $\mathbb{D}^2$-tight. It is simple to see that

$$\lim_{n\to\infty} P\left[\sup_{t\le 1} |Y_{\alpha,n}^{TSS}(t) - Y_\alpha^{TSS}(t)| > \delta\right] = 0$$

for all $\delta > 0$. Then we use again [17, Proposition 3-3] to obtain that the family of processes $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight.

The proof of both assertions 2 and 3 is similar to the one of [17, Théorème 3.5], therefore we omit it. ∎

In a similar way we obtain an analogous stability result if we replace the processes X_α^{TSS} and X_γ in equations (19) and (20) by X_α^{MTS} and X^{NIG} respectively and assumption **(H.2)** by

(H.2′) The family a_α (resp. h_α) converge uniformly to a (resp. h) on each compact set in $\mathbb{R}$, as $\alpha \to 1/2$.

We state this in the following theorem.

Theorem 4.3. *Under the assumptions **(H.1)** and **(H.2′)** we have*

1. *The family of processes $(Z_\alpha^{MTS}, X_\alpha^{MTS})$ with*

$$dZ_\alpha^{MTS}(t) = a_\alpha(Z_\alpha^{MTS}(t))\, dX_\alpha^{MTS}(t) + h_\alpha(Z_\alpha^{MTS}(t))\, dt, \quad Z_\alpha^{MTS}(0) = 0, \tag{22}$$

   is $\mathbb{D}^2$-tight.

2. *Any limit point (Z, X^{NIG}) of the family $(Z_\alpha^{MTS}, X_\alpha^{MTS})$ satisfies equation*

$$dZ(t) = a(Z(t))\, dX^{NIG}(t) + h(Z(t))\, dt, \quad Z(0) = 0. \tag{23}$$

3. *If uniqueness in law holds for equation (23), then*

$$(Z_\alpha^{MTS}, X_\alpha^{MTS}) \xrightarrow{\mathbb{D}^2} (Z, X^{NIG}), \quad \alpha \to 1/2.$$

4.2 The renormalized case

Finally, we conclude the section presenting a stability result for SDEs driving by the renormalized families $\{Y_\varepsilon,\, \tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$. To do so, let us consider the following equations

$$dZ_\varepsilon(t) = a_\varepsilon(Z_\varepsilon(t))\, d\tilde{Y}_\varepsilon(t) + h_\varepsilon(Z_\varepsilon(t)) dY_\varepsilon(t), \quad Z_\varepsilon(0) = 0, \tag{24}$$

and

$$dZ(t) = a(Z(t))\, dW(t) + h(Z(t))\, dt, \quad Z(0) = 0, \tag{25}$$

Our stability result then is stated in the following theorem.

Theorem 4.4. *Assume that*

(i) $\mu(\varepsilon)/\varepsilon \to +\infty$, $\varepsilon \to 0$;

(ii) $\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$, $\varepsilon \to 0$;

(iii) *the families $\{Y_\varepsilon,\, \varepsilon \in (0,1)\}$ and $\{\tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$ are independent;*

(iv) *the coefficients h_ε, a_ε and h, a satisfy the assumptions **(H.1)-(H.2)**.*

Then we have

1. *The family $\{(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon),\ \varepsilon \in (0,1)\}$ is $\mathbb{C}^3$-tight.*

2. *Any limit point $(Z, W, \mathbf{t})$ of the family $(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon)$ satisfies equation (25).*

3. If uniqueness in law holds for equation (25) then

$$(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon) \xrightarrow{\mathbb{C}^3} (Z, W, \mathbf{t}), \quad \varepsilon \to 0.$$

Proof. First we know that $\{Y_\varepsilon,\, \varepsilon \in (0,1)\}$ (resp. $\{\tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$) is a family of increasing processes (resp. martingales) which converges to the continuous increasing process $\mathbf{t}$ (resp. to the continuous martingale W). Since the two families are independents, then we have the following weak convergence

$$(Y_\varepsilon, \tilde{Y}_\varepsilon) \xrightarrow{\mathbb{C}} (\mathbf{t}, W), \quad \varepsilon \to 0.$$

Secondly, it is known that under (iv) equations (24) and (25) admit a weak solutions, see [11, Theorem 1.8]. Using the fact that $\sigma(\varepsilon)/\varepsilon \longrightarrow \infty$ as $\varepsilon \to 0$, we have

$$\int_{|s|>1} |s| d\tilde{\Lambda}_\varepsilon(s) = (\sigma(\varepsilon))^{-1} \int_{\sigma(\varepsilon)<|s|\le\varepsilon} |s| d\Lambda(s) \longrightarrow 0.$$

Finally the assumption **(H.1)** is sufficient for the continuity in the Skorohod space, cf. [15, Example 5.3]. So the assertions 1-3 follow from [17, Théorème 2.10]. □

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The α-dependence of stochastic differential equations driven by variants of α-stable processes

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Abstract

In this paper we investigate two variants of α-stable processes, namely tempered stable subordinators and modified tempered stable process as well as their renormalization. We study the weak convergence in the Skorohod space and prove that they satisfy the uniform tightness condition. Finally, applications to the α-dependence of the solutions of SDEs driven by these processes are discussed.

Keywords: Lévy processes, Uniform tightness, Skorohod space, Weak convergence, SDEs.

1 Introduction

In the last years, Lévy processes have received a great deal of attention fuelled by numerous applications. First of all, we would like to mention the stochastic finance theory, one of the principal subjects is the capital asset pricing model where the security price is allowed to have jumps, both big and small. Another reason to use models with jumps still in finance is for example in the stock market the price does not change continuously but change by units; the market is closed on weekends, holidays and opening prices often have jumps. We refer the interested reader to Cont and Tankov (2004) for an introduction to some financial models driven by Lévy processes. Second, in electrical engineering it is known that the telephone noise is non-Gaussian and the noise is modeled by a Lévy process. Indeed, the work of Stuck and Kleiner (1974)

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proposes to model the telephone noise by a stable process as well as a Lévy process with both jumps and a Wiener component. The latter model is suggested by the different sources of noise, specifically thermal noise corresponds to the Wiener part and the jump term comes from possibly thunderstorm. As a third example where a stochastic differential equation driven by a Lévy process appears we mention the model of an infinite capacity dam subject to an additive input process and a general release rule. The dynamics of the content of the dam is given by

$$dX_t = r(X_t)dt + dZ_t, \tag{1}$$

where Z is a Lévy process with nonnegative increments, $r(x)$ the release rate when the dam content is x. It has been suggested using empirical data that the Lévy measure of Z is the gamma measure, cf. Moran (1969), see also Protter and Talay (1997) for the numerical schemes of such models. When Z stands for a NIG process, then equation (1) was proposed as a generalized Hull-White model in finance, see Hainaut and MacGilchrist (2010).

In this paper we study the α-dependence of the solutions of the stochastic differential equations (SDEs) driven by variants of α-stable processes.

Firstly, we consider the classes of tempered stable subordinators X_α^{TSS} and modified tempered stable processes X_α^{MTS} with $\alpha \in (0, 1/2)$ and prove the weak convergence, in the Skorohod space endowed with the Skorohod topology, of X_α^{TSS} (resp. X_α^{MTS}) to the gamma process when $\alpha \to 0$ (resp. the normal inverse Gaussian process X^{NIG} when $\alpha \to 1/2$). The family $\{X_\alpha^{MTS}, \alpha \in (0, 1/2)\}$ was considered in Rachev *et al.* (2009) to develop the GARCH option price model. We want also to point out that the weak convergence of X_α^{TSS} to the gamma process was first established in Tsilevich *et al.* (2001) using other considerations. Indeed it is proved that the gamma process has been obtained as weak limit of renormalized stable processes. The family of the renormalized stable processes we identify as the family of tempered stable processes $\{X_\alpha^{TSS},\ \alpha \in (0, 1/2)\}$, see Remark 2.2 below. Moreover, we would like also to mention the work of Rydberg (1997) where an approximation of the NIG process, based on an appropriate discretization of the Lévy measure, was discussed. In this paper, instead we use the modified tempered stable process as an approximation of the NIG process.

The continuous dependence consists in investigating the conditions under which the solutions converge weakly. However, it is well known that the weak convergence is not sufficient to ensure the convergence of stochastic integral , see Kurtz and Protter (1996) and references therein. Among the sufficient conditions we cite the uniform tightness (**UT**), introduced by Stricker (1985). It should be noted that this condition has been used extensively to establish the results of stability of stochastic differential equations since its introduction, see for example Jacod (2004), Jacod and Protter (1998), Jakubowski *et al.* (1989), Kurtz and Protter (1996) and Mémin and Słomiński (1991). Thus we show that both driven fam-

ilies $\{X_\alpha^{TSS},\, X_\alpha^{MTS},\, \alpha \in (0,1/2)\}$ satisfy the (**UT**) condition. This allows us to establish the continuous dependence result of SDEs driven by these families.

Secondly, it is proven in Asmussen and Rosiński (2001) that the standard Brownian motion $\{W(t),\, t \in [0,1]\}$ is obtained as a weak limit, in the Skorohod space equipped with the uniform metric, of a suitable renormalization of certain classes of Lévy process which includes the family $\{X_\alpha^{MTS},\, \alpha \in (0,1/2\}$. More precisely, the Brownian motion W can be approximated by an appropriate renormalization of the compensated sum of small jumps of a given Lévy process, see Proposition 3.5 below. In the same spirit we mention the work Covo (2009) which completes, in some sense the previous one, where it is shown that the process $\{t,\, t \in [0,1]\}$ is a weak limit of a renormalized (in an appropriate sense) sum of small jumps of classes of subordinator. We note that the family $\{X_\alpha^{TSS},\, \alpha \in (0,1/2\}$ is among those classes. These two results lead us to consider the dependence problem of SDEs driven by these renormalized processes. The main tools we use to prove the continuous dependence result are the uniform tightness of the renormalized families and the stability of SDEs established in Mémin and Słomiński (1991).

2 Lévy processes and infinite divisibility

We start by recalling a few well-known facts about infinitely divisible distributions. We consider a class of Borel measures on $\mathbb{R}$ satisfying the following conditions:

$$\Lambda(\{0\}) = 0, \tag{2}$$

$$\int_{-\infty}^{+\infty} (s^2 \wedge 1) d\Lambda(s) < \infty. \tag{3}$$

This class will be denoted by $\mathfrak{M}$.

De Finetti (1929) introduced the notion of an infinitely divisible distribution and showed that they have an intimate relationship with Lévy processes. By the Lévy-Kintchine formula, all infinitely divisible distributions F_Λ are described via their characteristic function:

$$\phi_\Lambda(u) = \int_{-\infty}^{+\infty} e^{iux} dF_\Lambda(x) = e^{\Psi_\Lambda(u)},\ u \in \mathbb{R},$$

where the characteristic exponent Ψ_Λ, is given as

$$\Psi_\Lambda(u) = ibu - \frac{1}{2}cu^2 + \int_{-\infty}^{+\infty} (e^{ius} - 1 - ius\, 1\!\!1_{\{|s|<1\}}(s))\, d\Lambda(s),$$

where $b \in \mathbb{R}, c \geq 0$.

We assume as given a filtered probability space $(\Omega,\, \mathcal{F},\, P,\, (\mathcal{F}_t)_{t\in[0,1]})$ satisfying the usual hypothesis. A Lévy process $X = \{X(t), t \in [0,1]\}$ has the property

$$\mathbb{E}(e^{iuX(t)}) = e^{t\Psi(u)},\ t \in [0,1]\,,\ u \in \mathbb{R},$$

where $\Psi(u)$ is the characteristic exponent of $X(1)$ which has an infinitely divisible distribution. Thus, any infinitely divisible distribution F_Λ generates in a natural way a Lévy process X by setting the law of $X(1)$, $\mathcal{L}(X(1)) = F_\Lambda$. The three quantities (b, c, Λ) determine the law $\mathcal{L}(X(1))$. Since the distribution of a Lévy process $X = \{X(t), t \in [0,1]\}$ is completely determined by the marginal distribution $\mathcal{L}(X(1))$, and thus the process X itself completely. The measure Λ is called the Lévy measure whereas (b, c, Λ) is called the Lévy-Khintchine triplet.

Let us now give some examples of Lévy processes which will be used later on. We will present three classes related to the sample path properties. Namely subordinators, processes with paths of finite and infinite variations.

2.1 Subordinators

A subordinator is a one-dimensional increasing Lévy process starting from 0. Subordinators form one of the simplest family of Lévy processes. The law of a subordinator is specified by the Laplace transform of its one dimensional distributions. We assume throughout this paper that these processes have no drift.

We consider a subclass in $\mathfrak{M}$ of measures supported on $\mathbb{R}_+$ satisfying the following

$$\Lambda(0,\infty) \quad = \quad \infty, \tag{4}$$

$$\int_0^1 s \, d\Lambda(s) \quad < \quad \infty. \tag{5}$$

Any Lévy measure Λ satisfying conditions (4) and (5) generates a subordinator X, see for example (Bertoin, 1999, Theorem 1.2). We can therefore give its Laplace transform

$$\psi_\Lambda(u) := \mathbb{E}(e^{-uX(1)}) = \exp\left(\int_0^\infty (e^{-su} - 1) \, d\Lambda(s)\right), \quad u \in \mathbb{R}_+.$$

Remark 2.1. **(i)** *When X is a subordinator, the Laplace transform of its marginal distributions is much more useful, for both theoretical and practical applications, than the characteristic function.*

(ii) *The assumption (4) implies that the process X has infinite activity, that is, almost all paths have infinitely many jumps along any time interval of finite length. Whereas the condition (5) guarantees that almost all paths of X have finite variation.*

Examples

1. **Gamma process**. Consider the Lévy measure Λ_γ with density with respect to the Lebesgue measure defined by

$$d\Lambda_\gamma(s) := \frac{e^{-s}}{s} \mathbb{1}_{\{s>0\}} ds.$$

Then the corresponding process is known as gamma process. A simple calculation shows that

$$\psi_{\Lambda_\gamma}(u) = \frac{1}{1+u},$$

and the Laplace transform of the corresponding process has the form

$$\mathbb{E}_{\mu_\gamma}\left(e^{-uX_\gamma(t)}\right) = \exp\left(-t\log(1+u)\right) = \frac{1}{(1+u)^t},\ t \in [0,1].$$

Here μ_γ denotes the law of $X_\gamma(1)$.

2. **Stable subordinator (SS)**. Let $\alpha \in (0,1)$ be given and let Λ_α^{SS} be the Lévy measure given by

$$d\Lambda_\alpha^{SS}(s) := \frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{s^{1+\alpha}}1\!\!1_{\{s>0\}}ds.$$

Then we have

$$\psi_\Lambda(u) = \exp\left(-u^\alpha\right),$$

and

$$\mathbb{E}_{\mu_\alpha^{SS}}\left(e^{-uX_\alpha^{SS}(t)}\right) = \exp\left(-tu^\alpha\right),\ t \in [0,1].$$

Here μ_α^{SS} denotes the law of $X_\alpha^{SS}(1)$.

3. **Tempered stable subordinator (TSS).** A tempered stable subordinator is obtained by taking a stable subordinator and multiplying the Lévy measure by an exponential function, that is, an exponentially tempered version of the stable subordinator. More precisely, for $\alpha \in (0,1)$, we consider the Lévy measure

$$d\Lambda_\alpha^{TSS}(s) := \frac{1}{\alpha}e^{-s}d\Lambda_\alpha^{SS}(s) = \frac{1}{\Gamma(1-\alpha)}\frac{e^{-s}}{s^{1+\alpha}}1\!\!1_{\{s>0\}}ds. \tag{6}$$

Then we have

$$\psi_{\Lambda_\alpha^{TSS}}(u) = \exp\left(\frac{1-(1+u)^\alpha}{\alpha}\right)$$

and

$$\mathbb{E}_{\mu_\alpha^{TSS}}\left(e^{-uX_\alpha^{TSS}(t)}\right) = \exp\left(-t\frac{1-(1+u)^\alpha}{\alpha}\right),\ t \in [0,1].$$

Now let us give a concrete realization of a subordinator due to Tsilevich *et al.* (2001). We denote by

$$D = \left\{\eta = \sum z_i\delta_{x_i},\ x_i \in [0,1],\ z_i \in \mathbb{R}_+,\ \sum |z_i| < \infty\right\}$$

the real linear space of all finite real discrete measures in $[0,1]$. We define the coordinate process $\{X(t), t \in [0,1]\}$ on D by

$$X(t) : D \longrightarrow \mathbb{R}_+,\ \eta \mapsto X(t)(\eta) := \eta([0,t]), \quad t \in [0,1]$$

and $\mathcal{F}_t := \sigma(X(s),\ s \le t)$ denotes its own filtration.

Let Λ be a Lévy measure satisfying conditions (4) and (5) and μ_Λ be a probability measure on $(D, \mathcal{F}_1)$ with Laplace transform given by

$$\mathbb{E}_{\mu_\Lambda}\left(\exp\left(-\int_0^1 f(t)\, d\eta(t)\right)\right) = \exp\left(\int_0^1 \log(\psi_\Lambda(f(t))\, dt\right).$$

Here f is an arbitrary non-negative bounded Borel function on $[0,1]$. In particular, when $f(s) = u 1\!\!1_{[0,t]}(s)$, $u > 0$, $t \in (0,1]$ the Laplace transform of $X(t)$ is given by

$$\mathbb{E}_{\mu_\Lambda}(e^{-uX(t)}) = \exp\left(t \log(\psi_\Lambda(u))\right), \ t \in [0,1].$$

We call the pair (X, μ_Λ) a realization of a Lévy process with Lévy measure Λ which is a subordinator, cf. (Tsilevich *et al.*, 2001, Remark 2.1).

Now we would like to highlight the link between tempered stable and stable subordinators. First of all it follows from (6) that the Lévy measures Λ_α^{TSS} and Λ_α^{SS} are equivalent. Then we obtain from (Sato, 1999, Theorem 33.1) that X_α^{SS} and X_α^{TSS} have equivalent laws with density given in (Sato, 1999, Theorem 33.2), see (7) below. We notice that the authors in Tsilevich *et al.* (2001); Vershik and Yor (1995) constructed a family of measures, equivalent to α-stable laws with given densities which converges weakly to the gamma measure. This is the content of the following remark.

Remark 2.2. *Let $\tilde{X}_\alpha$ be a process such that the law $\tilde{\mu}_\alpha := \mathcal{L}(\tilde{X}_\alpha(1))$ is equivalent to μ_α^{SS} with density*

$$\frac{d\tilde{\mu}_\alpha}{d\mu_\alpha^{SS}}(\eta) = \frac{\exp(-\alpha^{-1/\alpha} X(1)(\eta))}{\mathbb{E}_{\mu_\alpha^{SS}}\left(\exp(-\alpha^{-1/\alpha} X(1)(\eta))\right)} = e^{\alpha^{-1}} e^{-\alpha^{-1/\alpha} X(1)(\eta)}. \tag{7}$$

Then the law of the tempered stable subordinator X_α^{TSS} is nothing but the law of the process $\alpha^{-1/\alpha}\tilde{X}_\alpha$.

2.2 Lévy processes with finite variation paths

We consider a Lévy process with the following triplet $(0, 0, \Lambda)$. We are interested here in the subclass of $\mathfrak{M}$ satisfying

$$\Lambda(\mathbb{R}) = \infty, \tag{8}$$

$$\int_{|s|\le 1} |s|\, d\Lambda(s) < +\infty. \tag{9}$$

Condition (9) means that the corresponding Lévy process has finite variation paths.

Examples

1. **Stable process (S)**. Symmetric α-stable processes X_α^S, with $\alpha \in (0,1)$, are the class of Lévy processes whose characteristic exponents correspond to those of symmetric α-stable distributions. The

corresponding Lévy measure is given by

$$d\Lambda_\alpha^S(s) = \left(\frac{1}{|s|^{1+\alpha}} 1\!\!1_{\{s<0\}} + \frac{1}{s^{1+\alpha}} 1\!\!1_{\{s>0\}} \right) ds.$$

The characteristic exponent $\Psi_{\Lambda_\alpha^S}$ has the form

$$\Psi_{\Lambda_\alpha^S}(u) = -|u|^\alpha, \; u \in \mathbb{R}.$$

2. **Tempered stable process (TS)**. It is well known that α-stable distributions, with $\alpha \in (0,1)$, have infinite p-th moments for all $p \geq \alpha$. This is due to the fact that its Lévy density decays polynomially. Tempering the tails with the exponential rate is one choice to ensure finite moments. The tempered stable distribution is then obtained by taking a symmetric α-stable distribution and multiplying its Lévy measure by an exponential functions on each half of the real axis. In explicit

$$d\Lambda_\alpha^{TS}(s) = \left(\frac{e^{-|s|}}{|s|^{1+\alpha}} 1\!\!1_{\{s<0\}} + \frac{e^{-s}}{s^{1+\alpha}} 1\!\!1_{\{s>0\}} \right) ds.$$

The characteristic exponent $\Psi_{\Lambda_\alpha^{TS}}$ is given by

$$\Psi_{\Lambda_\alpha^{TS}}(u) = \Gamma(-\alpha)[(1-iu)^\alpha + (1+iu)^\alpha - 2], \; u \in \mathbb{R}.$$

The associated Lévy process will be called tempered stable process and denoted by X_α^{TS}.

3. **Modified tempered stable process (MTS).**

The MTS distribution is obtained by taking an α-stable law with $\alpha \in (0, 1/2)$ and multiplying the Lévy measure by a modified Bessel function of the second kind on each side of the real axis. It is infinitely divisible and has finite moments of all orders. It behaves asymptotically like the 2α-stable distribution near zero and like the TS distribution on the tail. Then the Lévy density is given by

$$d\Lambda_\alpha^{MTS}(s) = \frac{1}{\pi} \left(\frac{K_{\alpha+\frac{1}{2}}(|s|)}{|s|^{\alpha+\frac{1}{2}}} 1\!\!1_{\{s<0\}} + \frac{K_{\alpha+\frac{1}{2}}(s)}{s^{\alpha+\frac{1}{2}}} 1\!\!1_{\{s>0\}} \right) ds.$$

$K_{\alpha+\frac{1}{2}}$ is the modified Bessel function of the second kind given by the following integral representation

$$K_{\alpha+\frac{1}{2}}(s) = \frac{1}{2} \left(\frac{s}{2} \right)^{\alpha+\frac{1}{2}} \int_0^{+\infty} \exp\left(-t - \frac{s^2}{4t} \right) t^{-\alpha-\frac{3}{2}} \, dt. \tag{10}$$

The characteristic exponent has the form

$$\Psi_{\Lambda_\alpha^{MTS}}(u) = \frac{1}{\sqrt{\pi}} 2^{-\alpha-\frac{1}{2}} \Gamma(-\alpha)[(1+u^2)^\alpha - 1], \; u \in \mathbb{R}.$$

The induced Lévy process, denoted by X_α^{MTS}, will be called modified tempered stable process. For additional details on MTS distributions the reader may consult Rachev *et al.* (2009).

2.3 Lévy process of infinite variation paths

Finally, we would like to consider a subclass of $\mathfrak{M}$ satisfying (8) and the following condition

$$\int_{|s|\leq 1} |s| \; d\Lambda(s) = \infty. \tag{11}$$

Examples

1. Symmetric α-stable processes, tempered stable processes, with $\alpha \in (1,2)$ and modified tempered stable processes, with $\alpha \in (1/2,1)$.

2. **Normal inverse Gaussian process (NIG).** The NIG distribution was introduced in finance by Barndorff-Nielsen. It might be of interest to know that the NIG distribution is a special case of the generalized hyperbolic distribution, introduced also by Barndorff-Nielsen to model the logarithm of particle size, see references below.

   Let $\{X^{NIG}(t),\, t \in [0,1]\}$ be a Lévy process with Lévy measure given by

   $$d\Lambda^{NIG}(s) = \frac{K_1(|s|)}{\pi|s|} ds,$$

   where K_1 is the modified Bessel function of the second kind with index 1. The characteristic exponent is equal to

   $$\Psi_{\Lambda^{\mathrm{NIG}}}(u) = \left(1 - \sqrt{1+u^2}\right), u \in \mathbb{R}.$$

   The process $\{X^{NIG}(t),\, t \in [0,1]\}$ is a Lévy process with the triplet $(0, 0, \Lambda^{NIG})$.

   For further results related to the normal inverse Gaussian distributions see Barndorff-Nielsen (1997, 1998) and Rydberg (1997, 1999).

We conclude this section with the following remark.

Remark 2.3. *All Lévy processes considered before are such that:*

1. *Their paths belong to the set of all càdlàg functions, denoted by* $\mathbb{D}([0,1],\mathbb{R})$*, i.e. all real-valued right continuous with left limits functions on* $[0,1]$.

2. *They are pure jump semimartingales processes without fixed times of discontinuity.*

3 Weak convergence and uniform tightness

In this section at first we present a result on weak convergence of the above families in $\mathbb{D}([0,1],\mathbb{R})$ endowed with the Skorohod topology $\mathcal{J}_1$, $(\mathbb{D},\, \mathcal{J}_1)$. This convergence will be denoted by "$\xrightarrow{\mathbb{D}}$". On the other hand,

since we will deal with continuous limit processes, we are interested in the tightness and weak convergence in the space $\mathbb{D}([0,1],\mathbb{R})$ equipped with the uniform topology $\mathcal{U}$, $(\mathbb{D},\mathcal{U})$. We will denoted them by "$\mathbb{C}$-tight" and "$\xrightarrow{\mathbb{C}}$", respectively. Finally, after recalling the definition of the uniform tightness as well as a useful criterion, we prove that the processes considered satisfy this condition, cf. Propositions 3.10 and 3.11 below.

3.1 Weak convergence in $(\mathbb{D},\mathcal{J}_1)$

In this subsection we present the weak convergence in $(\mathbb{D},\mathcal{J}_1)$ of the families of processes $\{X_\alpha^{TSS},\alpha\in(0,1/2)\}$ and $\{X_\alpha^{MSS},\ \alpha\in(0,1/2)\}$. We start with the following elementary lemma.

Lemma 3.1. *We have the following weak convergence of the one dimensional law:*

(i) $X_\alpha^{TSS}(1) \xrightarrow{\mathcal{L}} X_\gamma(1)$, $\alpha\to 0$.

(ii) $X_\alpha^{MTS}(1) \xrightarrow{\mathcal{L}} X^{NIG}(1)$, $\alpha\to 1/2$.

Proof. The result in (i) is a consequence of Proposition 6.3 in Tsilevich *et al.* (2001).
(ii) It is easy to see that the characteristic exponent $\Psi_{\Lambda_\alpha^{MTS}}(1)$ converge to $\Psi_{\Lambda^{NIG}}(1)$ when α goes to 1/2. This implies that $X_\alpha^{MTS}(1)$ converge weakly to $X^{NIG}(1)$. $\square$

Proposition 3.2. *We have the following weak convergence in* $(\mathbb{D},\mathcal{J}_1)$*:*

(i) $X_\alpha^{TSS} \xrightarrow{\mathbb{D}} X_\gamma$, *as* $\alpha\to 0$.

(ii) $X_\alpha^{MTS} \xrightarrow{\mathbb{D}} X^{NIG}$, *as* $\alpha\to 1/2$.

Proof. Since Lévy processes are semimartingales with stationary independent increments, then it follows from (Jacod and Shiryaev, 2003, Corollary 3.6) that the convergence of the marginal laws of $X_\alpha^{TSS}(1)$ and $X_\alpha^{MTS}(1)$ is equivalent to the weak convergence of processes X_α^{TSS} and X_α^{MTS} in $(\mathbb{D},\mathcal{J}_1)$. $\square$

Now we are interested in the weak convergence of certain renormalization of pure jump subordinator. Let X be a subordinator with Lévy measure Λ satisfying the conditions (4)-(5) and X_ε be the sum of its jumps of size in $(0,\varepsilon)$. Then the corresponding Lévy measure Λ_ε is nothing but the restriction of Λ to $(0,\varepsilon]$. We denote the expectation of $X_\varepsilon(1)$ by $\mu(\varepsilon) := \int_{(0,\varepsilon]} s d\Lambda(s)$. We consider the renormalized process $Y_\varepsilon := \mu(\varepsilon)^{-1}X_\varepsilon$ and state the following convergence result proved in Covo (2009).

Proposition 3.3. *The following statements hold, as* $\varepsilon\to 0$.

(i) *If* $\mu(\varepsilon)/\varepsilon\to c$, *where* $0<c<+\infty$, *then* $Y_\varepsilon \xrightarrow{\mathbb{D}} c^{-1}X_c^*$ *where* X_c^* *is a pure jump subordinator with Lévy measure given by* $d\Lambda_c^*(s)=\mathbb{1}_{(0,1]}(s)(c/s)ds$.

(ii) *If $\mu(\varepsilon)/\varepsilon \to +\infty$, then $Y_\varepsilon \xrightarrow{\mathbb{D}} \mathbf{t} := \{t, t \in [0,1]\}$.*

Remark 3.4. *Since Y_ε are Lévy processes and the limit process in the statement (ii) is continuous, then it follows from (Pollard, 1984, Theorem 19) that the convergence holds also in $(\mathbb{D}, \mathcal{U})$ as follows*

(ii)' *If $\mu(\varepsilon)/\varepsilon \to +\infty$, then $Y_\varepsilon \xrightarrow{\mathbb{C}} \mathbf{t}$.*

We give some examples of Lévy processes which illustrate the above proposition.

1. Gamma process, $\mu(\varepsilon)/\varepsilon \to 1$.

2. Stable and tempered stable subordinators, $\alpha \in (0,1)$, $\mu(\varepsilon)/\varepsilon \to +\infty$.

3.2 Weak convergence in $(\mathbb{D}, \mathcal{U})$

In this subsection we are interested in the weak convergence of certain renormalizations of Lévy processes. Let X be a Lévy process with characteristic function of the form

$$\mathbb{E}(e^{iuX(t)}) = \exp\left(t\left(ibu - \frac{1}{2}cu^2 + \int_{-\infty}^{+\infty}(e^{ius} - 1 - ius\,\mathbb{1}_{\{|s|<1\}}(s))\,d\Lambda(s)\right)\right)$$

where $t \in [0,1]$, $u \in \mathbb{R}$ and the Lévy measure Λ does not have atoms in some neighbourhood of the origin. For each $\varepsilon \in (0,1)$, let us consider $\tilde{X}_\varepsilon$ the compensated sum of jumps of X taking values in $(-\varepsilon, \varepsilon)$. It is well known that $\{\tilde{X}_\varepsilon, 0 < \varepsilon \le 1\}$ is a family of Lévy processes with characteristic function

$$\mathbb{E}(e^{iu\tilde{X}_\varepsilon(t)}) = \exp\left(t\int_{|s|\le\varepsilon}(e^{ius} - 1 - ius)\,d\Lambda(s)\right), \quad t \in [0,1].$$

It is clear that, for each $\varepsilon > 0$, $\tilde{X}_\varepsilon$ is a martingale with jumps bounded by ε with $\mathbb{E}(\tilde{X}_\varepsilon(1)) = 0$ and

$$\mathbb{E}(\tilde{X}_\varepsilon^2(1)) = \int_{|s|\le\varepsilon} s^2 d\Lambda(s) =: \sigma^2(\varepsilon).$$

We consider the renormalization process $\tilde{Y}_\varepsilon := \sigma(\varepsilon)^{-1}\tilde{X}_\varepsilon$ and state the following convergence result due to Asmussen and Rosiński (2001).

Proposition 3.5. *The following are equivalent*

1. *$\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$ as $\varepsilon \to 0$, where W is a standard Brownian motion.*
2. *$\dfrac{\sigma(\varepsilon)}{\varepsilon} \longrightarrow \infty$ as $\varepsilon \to 0$.*

Remark 3.6. *For each $\varepsilon \in (0,1)$, $\tilde{Y}_\varepsilon$ is a Lévy process with characteristic function given by*

$$\mathbb{E}(e^{iu\tilde{Y}_\varepsilon(t)}) = \exp\left(t\left[iub_\varepsilon + \int_{\mathbb{R}}\left(e^{ius} - 1 - ius\mathbb{1}_{\{|s|\le1\}}(s)\right)\,d\tilde{\Lambda}_\varepsilon(s)\right]\right), \ t \in [0,1],$$

where the Lévy measure $\tilde{\Lambda}_\varepsilon$ is defined, for any $B \in \mathcal{B}(\mathbb{R})$, by

$$\tilde{\Lambda}_\varepsilon(B) := \Lambda(\sigma(\varepsilon)B \cap (-\varepsilon, \varepsilon)), \tag{12}$$

and

$$b_\varepsilon := -\sigma(\varepsilon)^{-1} \int_{\sigma(\varepsilon)\wedge\varepsilon \le |s| \le \varepsilon} s \, d\Lambda(s). \tag{13}$$

We give some examples of Lévy processes for which the above renormalization converge.

1. Symmetric α-stable processes, $\alpha \in (0, 2)$, $\sigma(\varepsilon) = (2/(2-\alpha))^{1/2}\varepsilon^{1-\alpha/2}$.

2. Tempered stable processes, $\alpha \in (0, 1)$, $\sigma(\varepsilon) \ge (2/(2-\alpha))^{1/2}\varepsilon^{1-\alpha/2}e^{-\varepsilon/2}$.

3. Modified tempered stable processes, $\alpha \in (0, 1/2)$, $\sigma(\varepsilon) \approx (2/((2-2\alpha)\pi))^{1/2}\varepsilon^{1-\alpha}$.

4. Normal inverse Gaussian, $\sigma(\varepsilon) \approx (2/\pi)^{1/2}\varepsilon^{1/2}$.

We notice that the examples 1. and 4. above were considered in Asmussen and Rosiński (2001).

3.3 Uniform tightness of Lévy processes

First we recall the definition and criterion of the uniform tightness (**UT**) needed later on. The following definition was proposed by Jakubowski *et al.* (1989).

Definition 3.7. *A sequence of semimartingales $\{Z^n, n \ge 1\}$ is said to be uniformly tight if for each $t \in (0, 1]$, the set*

$$\left\{ \int_0^t H^n(s_-) \, dZ^n(s), H^n \in \mathcal{H}, n \ge 1 \right\}$$

is stochastically bounded (uniformly in n).

In the above definition $\mathcal{H}$ denotes the collection of simple predictable processes of the form

$$H(t) = H_0 + \sum_{i=1}^m H_i \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where H_i is $\mathcal{F}_{t_i}$-measurable such that $|H_i| \le 1$ and $0 = t_0 \le \ldots \le t_{m+1} = t$ is a finite partition of $[0, t]$.

In practice it is not easy to verify the (**UT**) condition as stated in Definition 3.7. Thus we look for a more convenient criterion due to Kurtz and Protter (1996). Let Z be an adapted process with càdlàg paths and $\{Z^n, n \in \mathbb{N}\}$ be a sequence of semimartingales, with the canonical decompositions

$$Z^n(t) = M^n(t) + A^n(t), \tag{14}$$

where A^n is a predictable process with locally bounded variation and M^n is a (locally bounded) local martingale.

Proposition 3.8. *[cf. Kurtz and Protter (1996)] Assume that $Z^n \xrightarrow{\mathbb{D}} Z$ and one of the following two conditions holds*

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left([M^n, M^n](1) + \int_0^1 |dA^n(t)|\right)\right\} < +\infty, \tag{15}$$

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left(\sup_{t\le 1}|\Delta M^n(t)| + \int_0^1 |dA^n(t)|\right)\right\} < +\infty. \tag{16}$$

*Then $\{Z^n, n\in\mathbb{N}\}$ satisfies (**UT**).*

Remark 3.9. 1. *If Z is a continuous semimartingale then we assume that $Z^n \xrightarrow{\mathbb{C}} Z$.*

2. *The conditions (15) and (16) imply the uniform controlled variation (**UCV**) of $\{Z^n, n\in\mathbb{N}\}$ introduced in Kurtz and Protter (1996).*

3. *Since $Z^n \xrightarrow{\mathbb{D}} Z$ then the (**UT**) and (**UCV**) are equivalent, see Kurtz and Protter (1996).*

Next, we are interested in the decomposition (14) for a Lévy process Z. We start by splitting Z into two parts depending on the size of the jumps:

$$Z(t) = R(t) + N(t)$$

with $N(t) = \sum_{s\le t}\Delta Z(s)\ 1\!\!1_{\{|\Delta Z(s)|>1\}}$ and R with jumps bounded by 1. Since R is a Lévy process with bounded jumps its canonical decomposition is, by means of (Applebaum, 2004, pp. 103), of the simple form $R(t) = R_0(t) + t\mathbb{E}(R(1))$ where $\{R_0(t) : t\in[0,1]\}$ is a càdlàg centred square-integrable martingale with jumps bounded by 1. Hence the decomposition (14) takes the form

$$Z(t) = R_0(t) + t\mathbb{E}(R(1)) + \sum_{s\le t}\Delta Z(s)\ 1\!\!1_{\{|\Delta Z(s)|>1\}}. \tag{17}$$

Now we are able to state the main result of this subsection.

Proposition 3.10. *The following families satisfy (**UT**)*

(i) $\{X_\alpha^{TSS},\, \alpha\in(0,1/2)\}$,

(ii) $\{X_\alpha^{MTS},\, \alpha\in(0,1/2)\}$.

Proof. Since the families $\{X_\alpha^{TSS},\, \alpha\in(0,1/2)\}$ and $\{X_\alpha^{MTS},\, \alpha\in(0,1/2)\}$ are weakly convergent, then in order to obtain the (**UT**) property, we have only to check condition (15) of Proposition 3.8.

(i) The decomposition (17) for the process X_α^{TSS} is given by

$$X_\alpha^{TSS}(t) = R_{\alpha,0}^{TSS}(t) + t\mathbb{E}(R_\alpha^{TSS}(1)) + \sum_{s\le t}\Delta X_\alpha^{TSS} 1\!\!1_{\{|\Delta X_\alpha^{TSS}|>1\}}. \tag{18}$$

Thus, condition (15) becomes

$$\sup_{\alpha\in(0,1/2)} \left(\int_0^1 s^2 \, d\Lambda_\alpha^{TSS}(s) + \int_0^{+\infty} s \, d\Lambda_\alpha^{TSS}(s) \right) < +\infty,$$

which is simple to verify.

(ii) It is easy to see that $\mathbb{E}(R_\alpha^{MTS}(1)) = 0$. Then the (**UT**) condition follows from

$$\sup_{\alpha\in(0,1/2)} \left(\int_{|s|\le 1} s^2 \, d\Lambda_\alpha^{MTS}(s) + \int_{|s|>1} |s| \, d\Lambda_\alpha^{MTS}(s) \right) < +\infty.$$

To show this we use the integral representation (11) for the Bessel function $K_{\alpha+1/2}$ and estimate the above integrals as

$$\begin{aligned}
\int_{|s|>1} |s| \, d\Lambda_\alpha^{MTS}(s) &= 2^{-\alpha-1/2} \int_1^{+\infty} \int_0^{+\infty} s \, e^{-\frac{s^2}{4t}} e^{-t} t^{-(\alpha+3/2)} dt ds \\
&= 2^{1/2-\alpha} \int_0^{+\infty} e^{-(t+\frac{1}{4t})} t^{-\alpha-1/2} dt \\
&\le 5 \, 2^{1/2-\alpha} \int_{1/4}^{+\infty} e^{-(t+\frac{1}{4t})} dt.
\end{aligned}$$

$$\begin{aligned}
\int_{|s|\le 1} s^2 \, d\Lambda_\alpha^{MTS}(s) &\le 2 \int_0^{+\infty} s^2 \, d\Lambda_\alpha^{MTS}(s) \\
&= \sqrt{\pi} \, 2^{1/2-\alpha} \, \Gamma(1-\alpha).
\end{aligned}$$

This completes the proof. $\square$

Next we state the (**UT**) property for the renormalized families $\{Y_\varepsilon, \, \varepsilon \in (0,1)\}$ and $\{\tilde{Y}_\varepsilon, \, \varepsilon \in (0,1)\}$.

Proposition 3.11. **(i)** *Assume that $\mu(\varepsilon)/\varepsilon$ converges in $(0,+\infty]$. Then the renormalized family $\{Y_\varepsilon, \, \varepsilon \in (0,1)\}$ satisfies (**UT**).*

(ii) *Assume that $\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$. Then the renormalized family $\{\tilde{Y}_\varepsilon, \, \varepsilon \in (0,1)\}$ satisfies (**UT**).*

Proof. (i) Since the process Y_ε is a pure jump subordinator, then the condition (15) becomes

$$\sup_{\varepsilon\in(0,1)} \left\{ \mathbb{E}\left(\int_0^1 |dY_\varepsilon(t)| \right) \right\} = \sup_{\varepsilon\in(0,1)} \mathbb{E}\left(Y_\varepsilon(1)\right) = 1. \tag{19}$$

So the (**UT**) condition is a consequence of Proposition 3.8.

(ii) First notice that, for each $\varepsilon \in (0,1)$, $\tilde{Y}_\varepsilon$ is a martingale with jumps bounded by $\varepsilon/\sigma(\varepsilon)$. Thus we obtain

$$\mathbb{E}\left(\sup_{t\le 1} |\Delta \tilde{Y}_\varepsilon(t)| \right) \le \frac{\varepsilon}{\sigma(\varepsilon)}.$$

As a consequence of statement 2 of Proposition 3.5 we have

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left(\sup_{t\leq 1}|\Delta\tilde{Y}_\varepsilon(t)|\right) < \infty,$$

which implies that condition (16) is satisfied. Since $\tilde{Y}_\varepsilon$ is weakly convergent, then (**UT**) condition follows from Proposition 3.8. □

4 α-Continuity of SDEs driven by Lévy processes

The previous section established the weak convergence and uniform tightness for certain families of Lévy processes. Now we would like to apply these results to study the continuous dependence problem for SDEs driven by these families of Lévy processes. For a survey on SDEs driven by Lévy processes we refer to Bass (2004). To begin, we give some notations useful in the sequel: for each $n \in \{2, 3, \ldots\}$, "$\stackrel{\mathbb{D}^n}{\longrightarrow}$" and "$\mathbb{D}^n$-tight" denote the weak convergence and tightness in $\mathbb{D}([0,1],\mathbb{R}^n)$ endowed with the Skorohod topology. In the same way "$\stackrel{\mathbb{C}^n}{\longrightarrow}$" and "$\mathbb{C}^n$-tight" denote the weak convergence and tightness for the uniform topology.

4.1 The modified tempered stable case

We will make the following assumptions

(H.1) $a_\alpha, h_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous such that $|a_\alpha(x)| + |h_\alpha(x)| \leq K(1+|x|)$ for all $\alpha \in (0, 1/2)$, $x \in \mathbb{R}$.

(H.2) The family a_α (resp. h_α) converge uniformly to a (resp. h) on each compact set in $\mathbb{R}$, as $\alpha \to 0$.

We consider the following SDEs

$$dY_\alpha^{TSS}(t) = a_\alpha(Y_\alpha^{TSS}(t_-))dX_\alpha^{TSS}(t) + h_\alpha(Y_\alpha^{TSS}(t))dt, \quad Y_\alpha^{TSS}(0) = 0, \tag{20}$$

and

$$dY(t) = a(Y(t_-))dX_\gamma(t) + h(Y(t))dt, \quad Y(0) = 0. \tag{21}$$

Remark 4.1. *1. Under the assumption **(H.1)**, for each $\alpha \in (0, 1/2)$, the equation (20) admits a weak solution, see Jacod and Mémin (1981).*

2. Since the coefficients a_α and a are not Lipschitz, then we do not have uniqueness of solutions for either equation (20) or equation (21).

The first α-continuity result concerns the class of tempered stable subordinators.

Theorem 4.2. *Under the assumptions **(H.1)-(H.2)** we have*

1. *The family of processes* $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ *is* $\mathbb{D}^2$*-tight.*

2. *Any limit point* (Y, X_γ) *of the family* $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ *satisfies equation (21).*

3. *If uniqueness in law holds for the equation (21), then*

$$(Y_\alpha^{TSS}, X_\alpha^{TSS}) \xrightarrow{\mathbb{D}^2} (Y, X_\gamma), \quad \alpha \to 0.$$

Proof. 1. At first we show that the family $\{Y_\alpha^{TSS},\, \alpha \in (0,1/2)\}$ verify the (**UT**) condition. For that we need to prove that the family $\{\sup_{s\in[0,1]} |Y_\alpha^{TSS}(s)|,\, \alpha \in (0,1/2)\}$ is bounded in probability. Indeed, as Y_α^{TSS} satisfies the equation (20) then, for $t \in [0,1]$, we have

$$Y_\alpha^{TSS}(t) = \int_0^t a_\alpha(Y_\alpha^{TSS}(s_-))dX_\alpha^{TSS}(s) + \int_0^t h_\alpha(Y_\alpha^{TSS}(s))ds. \tag{22}$$

The set $\left\{t : Y_\alpha^{TSS}(t) \neq Y_\alpha^{TSS}(t_-)\right\}$ is countable and so far it is Lebesgue negligible. Owing to this fact we may replace $h_\alpha(Y_\alpha^{TSS}(t))$ by $h_\alpha(Y_\alpha^{TSS}(t_-))$ in the right-hand side of (22) and obtain

$$Y_\alpha^{TSS}(t) = \int_0^t a_\alpha(Y_\alpha^{TSS}(s_-))dX_\alpha^{TSS}(s) + \int_0^t h_\alpha(Y_\alpha^{TSS}(s_-))ds.$$

Using assumption **(H.1)** we get

$$|Y_\alpha^{TSS}(t)| \leq K \int_0^t (1 + |Y_\alpha^{TSS}(s_-)|)d\{s + X_\alpha^{TSS}(s)\},\ \forall\, t \in [0,1].$$

It follows from a Gronwall type inequality, see (Protter, 2005, pp. 352), that

$$|Y_\alpha^{TSS}(t)| \leq K \exp(t + X_\alpha^{TSS}(t)),\ \forall\, t \in [0,1].$$

Now since X_α^{TSS} is increasing it yields

$$\sup_{t\in[0,1]} |Y_\alpha^{TSS}(t)| \leq K \exp(1 + X_\alpha^{TSS}(1)).$$

We infer the boundedness in probability of the family $\{\sup_{s\in[0,1]} |Y_\alpha^{TSS}(s)|,\, \alpha \in (0,1/2)\}$ from the boundedness in probability of the the family $\{X_\alpha^{TSS}(1),\, \alpha \in (0,1/2)\}$ which is a consequence of the uniform tightness of the family $\{X_\alpha^{TSS},\, \alpha \in (0,1/2)\}$, see Lemma 1.2 of Jakubowski *et al.* (1989) or Stricker (1985).

Hence the family $\{\sup_{s\in[0,1]} |a_\alpha(Y_\alpha^{TSS}(s))|,\, \alpha \in (0,1/2)\}$ (resp. $\{\sup_{s\in[0,1]} |h_\alpha(Y_\alpha^{TSS}(s))|,\, \alpha \in (0,1/2)\}$) is also bounded in probability since a_α (resp. h_α) has at most linear growth. Therefore it is easy to see that the family $\{\int_0^\cdot h_\alpha(Y_\alpha^{TSS}(t))dt,\, \alpha \in (0,1/2)\}$ satisfies the (**UT**) condition. On the other hand, the uniform tightness of the family $\{\int_0^\cdot a_\alpha(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t), \alpha \in (0,1/2)\}$ follows from (Mémin and Słomiński, 1991, Lemme 1-6). As a consequence we get the (**UT**) condition for the family $\{Y_\alpha^{TSS},\, \alpha \in (0,1/2)\}$.

On the next step we show that the family of processes $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight. Since the function a_α is continuous we can always find a sequence of C^2 functions, $\{a_{\alpha,n},\, n \in \mathbb{N}\}$, which approximate uniformly a_α on compact sets of $\mathbb{R}$. Now let us consider the sequence of process $Y_{\alpha,n}^{TSS}$ defined by

$$dY_{\alpha,n}^{TSS}(t) = a_{\alpha,n}(Y_\alpha^{TSS}(t_-))dX_\alpha^{TSS}(t) + h_\alpha(Y_\alpha^{TSS}(t))dt, \quad Y_\alpha^{TSS}(0) = 0. \tag{23}$$

As the function $a_{\alpha,n}$ is of class C^2 then we get from (Mémin and Słomiński, 1991, Lemme 1-7) that the family $\{a_{\alpha,n}(Y_\alpha^{TSS}), \alpha \in (0, 1/2)\}$ is uniformly tight. Now it follows from (Mémin and Słomiński, 1991, Proposition 3-3) (see also Kurtz and Protter (1991)) that the family of processes $(\int_0^\cdot a_{\alpha,n}(Y_\alpha^{TSS}(t))dX_\alpha^{TSS}(t), X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight and consequently $(Y_{\alpha,n}^{TSS}, X_\alpha^{TSS})$ is also $\mathbb{D}^2$-tight. It is simple to see that

$$\lim_{n\to\infty} P\left[\sup_{t\le 1} |Y_{\alpha,n}^{TSS}(t) - Y_\alpha^{TSS}(t)| > \delta\right] = 0$$

for all $\delta > 0$. Then we use again (Mémin and Słomiński, 1991, Proposition 3-3) to obtain that the family of processes $(Y_\alpha^{TSS}, X_\alpha^{TSS})$ is $\mathbb{D}^2$-tight.

The proof of both assertions 2 and 3 is similar to the one of (Mémin and Słomiński, 1991, Théorème 3.5), therefore we omit it. □

We now turn to an example of equation (21), for which there is no uniqueness in law.

Example 4.3. *We consider the following equation*

$$dY(t) = dX_\gamma(t) + h(Y(t))dt, \quad Y(0) = 0, \tag{24}$$

where h is bounded continuous and equal to $\operatorname{sign}(x)|x|^\beta$ *in some neighborhood of $x = 0$ for certain positive constant $\beta < 1$.*

This example is inspired by the work of of Tanaka et al. *(1974), who treats the problem of uniqueness of the equation (24) with the symmetric stable process instead of the gamma process. Precisely, the authors show that the equation (24), with the drift h given above, does not admit the uniqueness property. The key of the proof is the asymptotic rate of growth of the sample paths of the symmetric stable process at the origin. In our case the gamma process satisfies the following short time behavior*

$$\lim_{t\downarrow 0} \frac{X_\gamma(t)}{t^{1/(1-\beta)}} = 0 \quad a.s. \tag{25}$$

which is a consequence of the finiteness of $\int_0^1 \Lambda_\gamma[t^{1/(1-\beta)}, +\infty)\, dt$*, see Bertoin (1996) or Sato (1999). This is sufficient to show non-uniqueness proceeding along the lines as in Tanaka* et al. *(1974), Theorem 3.2.*

In a similar way we obtain an analogous α-continuity result if we replace the processes X_α^{TSS} and X_γ in equations (20) and (21) by X_α^{MTS} and X^{NIG} respectively and assumption **(H.2)** by

(H.2′) The family a_α (resp. h_α) converge uniformly to a (resp. h) on each compact set in $\mathbb{R}$, as $\alpha \to 1/2$.

We state this in the following theorem.

Theorem 4.4. *Under the assumptions **(H.1)** and **(H.2′)** we have*

1. *The family of processes $(Z_\alpha^{MTS}, X_\alpha^{MTS})$ with*

$$dZ_\alpha^{MTS}(t) = a_\alpha(Z_\alpha^{MTS}(t_-))\, dX_\alpha^{MTS}(t) + h_\alpha(Z_\alpha^{MTS}(t))\, dt, \quad Z_\alpha^{MTS}(0) = 0, \tag{26}$$

   is $\mathbb{D}^2$-tight.

2. *Any limit point (Z, X^{NIG}) of the family $(Z_\alpha^{MTS}, X_\alpha^{MTS})$ satisfies equation*

$$dZ(t) = a(Z(t_-))\, dX^{NIG}(t) + h(Z(t))\, dt, \quad Z(0) = 0. \tag{27}$$

3. *If uniqueness in law holds for equation (27), then*

$$(Z_\alpha^{MTS}, X_\alpha^{MTS}) \xrightarrow{\mathbb{D}^2} (Z, X^{NIG}), \quad \alpha \to 1/2.$$

4.2 The renormalized case

Finally, we conclude the section presenting a ε-continuity result for SDEs driving by the renormalized families $\{Y_\varepsilon, \tilde{Y}_\varepsilon, \varepsilon \in (0,1)\}$. To do so, let us consider the following equations

$$dZ_\varepsilon(t) = a_\varepsilon(Z_\varepsilon(t_-))\, d\tilde{Y}_\varepsilon(t) + h_\varepsilon(Z_\varepsilon(t)) dY_\varepsilon(t), \quad Z_\varepsilon(0) = 0, \tag{28}$$

and

$$dZ(t) = a(Z(t))\, dW(t) + h(Z(t))\, dt, \quad Z(0) = 0, \tag{29}$$

Our result then is stated in the following theorem.

Theorem 4.5. *Assume that*

(i) $\mu(\varepsilon)/\varepsilon \to +\infty$, $\varepsilon \to 0$;

(ii) $\tilde{Y}_\varepsilon \xrightarrow{\mathbb{C}} W$, $\varepsilon \to 0$;

(iii) *the families $\{Y_\varepsilon, \varepsilon \in (0,1)\}$ and $\{\tilde{Y}_\varepsilon, \varepsilon \in (0,1)\}$ are independent;*

(iv) *the coefficients h_ε, a_ε and h, a satisfy the assumptions **(H.1)-(H.2)**.*

Then we have

1. *The family $\{(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon),\ \varepsilon \in (0,1)\}$ is $\mathbb{C}^3$-tight.*

2. *Any limit point* $(Z, W, \mathbf{t})$ *of the family* $(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon)$ *satisfies equation (29).*

3. *If uniqueness in law holds for equation (29) then*

$$(Z_\varepsilon, \tilde{Y}_\varepsilon, Y_\varepsilon) \xrightarrow{\mathbb{C}^3} (Z, W, \mathbf{t}), \quad \varepsilon \to 0.$$

Proof. First we know that $\{Y_\varepsilon,\, \varepsilon \in (0,1)\}$ (resp. $\{\tilde{Y}_\varepsilon,\, \varepsilon \in (0,1)\}$) is a family of increasing processes (resp. martingales) which converges to the continuous increasing process $\mathbf{t}$ (resp. to the continuous martingale W). Since the two families are independents, then we have the following weak convergence

$$(Y_\varepsilon, \tilde{Y}_\varepsilon) \xrightarrow{\mathbb{C}} (\mathbf{t}, W), \quad \varepsilon \to 0.$$

Secondly, it is known that under (iv) equations (28) and (29) admit a weak solutions, see (Jacod and Mémin, 1981, Theorem 1.8). Using the fact that $\sigma(\varepsilon)/\varepsilon \longrightarrow \infty$ as $\varepsilon \to 0$, we have

$$\int_{|s|>1} |s| d\tilde{\Lambda}_\varepsilon(s) = (\sigma(\varepsilon))^{-1} \int_{\sigma(\varepsilon)<|s|\leq\varepsilon} |s| d\Lambda(s) \longrightarrow 0.$$

Finally the assumption **(H.1)** is sufficient for the continuity in the Skorohod space, cf. (Kurtz and Protter, 1991, Example 5.3). So the assertions 1-3 follow from (Mémin and Słomiński, 1991, Théorème 2.10). □

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